EQUILIBRIUM OF A MEMBRANE SHELL OF REVOLUTION AT LARGE DEFORMATIONS

(RAVNOVESIE BEZMOMENTNOI OBOLOCHKI VRASHCHENIIA PRI BOL'SHIKH DEFORMATSIIAKH)

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We consider in the following a membrane shell, with one or two rigid heads, in the state of equilibrium with large displacements and deformations; the shape of the shell in the unloaded state is defined by rotation of an arbitrary smooth contour, and it is supposed to be loaded by internal pressure, varying in axial direction, and by forces applied to the heads. The material is considered to be incompressible, and its mechanical properties are determined by the mutual dependence between the stresses and the true (logarithmic) deformations. In addition to the hypotheses, which represent the foundation for the relations connecting these quantities with each other (see [1,2]), the conventional assumptions are used, generally accepted in the theory of thin membrane shells. Similar problems were studied earlier (see bibliography in [3] and [4]) with some specific simplified formulations, like the case of inextensible material or the case of "equal strength", etc. We shall derive below the fundamental systems of equations, to the solution of which the problem is reducible, and the case of a shell of originally cylindrical shape will be considered in detail. This latter problem was discussed earlier for the case of uniform pressure [4]. References [5] to [7] deal, on the basis of the same fundamental assumptions, with the problem of deformation of a membrane loaded by uniform pressure.

1. Basic conditions and relationships. Two zones are formed in a shell in the most general case of equilibrium: a zone of tension and a zone of "folding" [3]. The former is characterized by appearance of positive principal curvatures and positive principal stresses, while in the latter the circumferential stress is to be considered equal to zero, the shell shows folds, and there arises some kind of a system of filaments under transverse forces and external tension.

Consider a shell (Fig. 1), referred to a system of dimensionless cylindrical coordinates $x\,\theta\,z$ invariably fixed at one of the apexes. Call ξ and η the values of x and y, respectively, for the undeformed shell. The relations connecting these coordinates with the corresponding dimensional coordinates X, Y, r, ζ shall be stated by means of the formulas

$$x = \frac{X}{R_1} A, \qquad y = \frac{Y}{R_1} A \qquad (1.1)$$
$$\xi = \frac{r}{R_1} A, \qquad \eta = \frac{\zeta}{R_1} A$$

where R_1 represents some characteristic initial dimension of the shell, while A is an arbitrary dimensionless parameter.

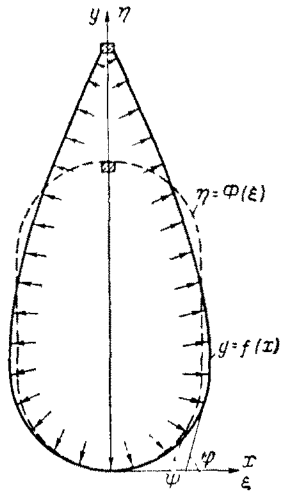

Fig. 1.

The initial shell profile shall be given by the formula

$$\eta = \Phi(\xi) \qquad (1.2)$$

The principal "true" extensions and shears are defined by

$$\varepsilon_1 = \ln(1 + e_1), \qquad \varepsilon_2 = \ln(1 + e_2), \qquad \varepsilon_3 = \ln(1 + e_3)$$
$$\gamma_{12} = \ln\frac{1 + e_1}{1 + e_2}, \qquad \gamma_{23} = \ln\frac{1 + e_2}{1 + e_3}, \qquad \gamma_{31} = \ln\frac{1 + e_3}{1 + e_1} \qquad (1.3)$$

where e_1, e_2, e_3 are the usual extension components. The principal shear stresses are

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}, \qquad \tau_{23} = \frac{\sigma_2 - \sigma_3}{2}, \qquad \tau_{31} = \frac{\sigma_3 - \sigma_1}{2} \qquad (1.4)$$

where σ_1 and σ_2 are the meridional and the circumferential stress, respectively.

The diagram of the "true" shear shall be approximated by the curve

$$\gamma = \operatorname{sign}\tau \left|\frac{\tau}{K}\right|^{1/\mu} \qquad (1.5)$$

where K and μ are constants derived from the conditions for the best approximation. Replacing γ by the principal shear of maximum magnitude and τ by the corresponding principal shear stress with σ_3 assumed to be zero, we find

$$p_1 = (\varepsilon_1 - \varepsilon_3)^\mu \quad \text{for } |p_1| \geqslant |p_2| \qquad \left(p_1 = \frac{\sigma_1}{2K}, \quad p_2 = \frac{\sigma_2}{2K}\right) \qquad (1.6)$$
$$p_2 = (\varepsilon_2 - \varepsilon_3)^\mu \quad \text{for } |p_2| \geqslant |p_1|$$

Note that if we assume $\mu = 1$ we will have to do with an elastic shell of incompressible material; the difference between the two cases disappears, p_1 and p_2 follow directly from (1.6), and K becomes the shear modulus.

Assuming that the principal shear stresses are proportional to the principal true shears, we find that

$$\frac{p_1}{p_2} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2 - \varepsilon_3} \tag{1.7}$$

The condition of incompressibility of the material gives

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \tag{1.8}$$

We may use instead of (1.5) an analogously approximated relationship between the stress intensity σ_i and the intensity ϵ_i of the "true" deformations

$$\varepsilon_i = \left(\frac{\sigma_i}{K_i}\right)^{1/\mu} \tag{1.9}$$

where K_i and μ are constants. Then replacing by p_1 and p_2 respectively, the quantities σ_1 and σ_2, referred to K_i, and using (1.7) instead of (1.6), we obtain

$$p_1 = \tfrac{2}{3}\,\varepsilon_i^{\mu-1}(\varepsilon_1 - \varepsilon_3), \qquad p_2 = \tfrac{2}{3}\,\varepsilon_i^{\mu-1}(\varepsilon_2 - \varepsilon_3) \tag{1.10}$$

Note that, inasmuch as $\sigma_3 = 0$ and the material is incompressible

$$\begin{gathered}\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} = \sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2} \\ \varepsilon_i = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_3-\varepsilon_1)^2} = \frac{2}{\sqrt{3}}\sqrt{\varepsilon_2^2+\varepsilon_3^2+\varepsilon_2\varepsilon_3}\end{gathered} \tag{1.11}$$

In this case we arrive at the relations valid for an elastic shell by putting K_i equal to the modulus of elasticity and μ equal to 1.

Denote by: S_1 the curvilinear coordinate measured along the arc of the meridian after deformation; S_{10} the same coordinate before deformation; H the thickness of the shell after deformation; and H_1 its value in the initial state. Then we will have in the zone of tension

$$e_1 = \frac{dS_1 - dS_{10}}{dS_{10}} = \frac{dX\cos\psi}{dr\cos\varphi} - 1 = \frac{dx}{d\xi}\frac{\cos\psi}{\cos\varphi} - 1$$

$$e_2 = \frac{X - r}{r} = \frac{x}{\xi} - 1, \qquad e_3 = \frac{H - H_1}{H_1} = h - 1$$

where ϕ is the angle between the tangent to the meridian curve and the plane normal to the axis of the shell, while ψ represents the value of the same angle for the initial shape of the shell and h is the latter's dimensionless thickness. The formulas for e_1 and e_3 can be used also in establishing expressions for the deformations in the folded zone. Ultimately, we obtain for the zone in tension

$$\varepsilon_1 = \ln\left(\frac{dx}{d\xi}\frac{\cos\psi}{\cos\varphi}\right), \qquad \varepsilon_2 = \ln\frac{x}{\xi}, \qquad \varepsilon_3 = \ln h \tag{1.12}$$

where, by virtue of (1.2), $\cos\psi$ is a given function of ξ.

2. Fundamental systems of equations. The equilibrium equations of a shell element in the zone of extension can be written, in the case of variable shell thickness and pressure, in terms of dimensionless coordinates and quantities, in the following manner:

$$\frac{d}{dx}(xhp_1) = p_2 h, \qquad \frac{d}{dx}(xhp_1\sin\varphi) = \frac{Q(y)}{A}x \qquad \left(Q(y) = \frac{R_1}{2KH_1}q(y)\right) \tag{2.1}$$

The function $q(y)$ represents here the pressure intensity. Taking into account that

$$\frac{dy}{dx} = \tan\varphi \tag{2.2}$$

we can derive from Equations (1.7), (1.8), (1.12), (2.1), taken in conjunction with one of the relations (1.6), a system of four differential equations of the first order for x, y, ϕ and h considered as functions of ξ at a given $Q(y)$. In the case of $p_1 \geqslant p_2$ the system just mentioned will be of the form

$$\begin{gathered}\frac{dx}{d\xi} = \frac{\xi}{xh}\frac{\cos\varphi}{\cos\psi}, \qquad \frac{dy}{d\xi} = \frac{\xi}{xh}\frac{\sin\varphi}{\cos\psi}\\ \frac{d\varphi}{d\xi} = \frac{\xi}{xh\cos\psi}\left(\ln\frac{\xi}{xh^2}\right)^{-1}\left[\frac{Q(y)}{Ah}\left(\ln\frac{\xi}{xh^2}\right)^{1-\mu} - \frac{\sin\varphi}{x}\ln\frac{x}{\xi h}\right]\\ \frac{dh}{d\xi} = \frac{\mu x^2 h\cos\psi - \xi^2[\mu + \ln(x^2h/\xi^2)]\cos\varphi}{x^2\xi[2\mu + \ln(xh^2/\xi)]\cos\psi}\end{gathered} \tag{2.3}$$

The dimensionless expressions for the stresses will be

$$p_1 = \left(\ln\frac{\xi}{xh^2}\right)^{\mu}, \qquad p_2 = \left(\ln\frac{\xi}{xh^2}\right)^{\mu-1}\ln\frac{x}{\xi h} \tag{2.4}$$

In the case of $p_2 \geqslant p_1$ the last two formulas of the system (2.3) are to be replaced by the following:

$$\begin{gathered}\frac{d\varphi}{d\xi} = \frac{\xi}{xh\cos\psi}\left(\ln\frac{\xi}{xh^2}\right)^{-1}\left[\frac{Q(y)}{Ah}\left(\ln\frac{x}{\xi h}\right)^{1-\mu} - \frac{\sin\varphi}{x}\ln\frac{x}{\xi h}\right]\\ \frac{dh}{d\xi} = \frac{x^2h\left(\mu\ln\frac{xh^2}{\xi} - 3\ln h\right)\cos\psi - \xi^2\left(\mu\ln\frac{xh^2}{\xi} - 3\ln h - \ln\frac{x^2h}{\xi}\ln\frac{x}{\xi h}\right)\cos\varphi}{x^2\xi\left[3\ln\frac{x}{\xi} + \left(\mu - \ln\frac{x}{\xi h}\right)\ln\frac{xh^2}{\xi}\right]\cos\psi}\end{gathered} \tag{2.5}$$

where

$$p_1 = \left(\ln \frac{x}{\xi h}\right)^{\mu-1} \ln \frac{\xi}{xh^2}, \qquad p_2 = \left(\ln \frac{x}{\xi h}\right)^{\mu} \tag{2.6}$$

Considering the folded zone, we introduce the concept of a certain defining surface, the surface which would be generated by a system of filaments under pressure, acting in the folded zone of the real shell. At one end this system of filaments absorbs the tension corresponding to the meridional stresses in the zone under extension at $p_2 = 0$, at the other end the filaments are attached to the rigid head of a given radius.

In the following we shall denote, with reference to the folded zone, by x and y the coordinates of the determining surface and by ϕ the angle between the tangent to its meridian curve and the plane normal to the axis of the shell. Disregarding the difference in curvatures of different meridians of the actual middle surface for one and the same y, considering the slope angles of their tangents to be equal to ϕ and assuming that the dimensionless thickness of the shell and the amount of the meridional stress do not vary with varying θ, we can obtain the fundamental system of equations for the determination of x, y, ϕ and h in terms of ξ, using the same original equations and relationships as in the zone under extension. To this end we have to take

$$p_2 = 0, \quad \varepsilon_2 - \varepsilon_3 = 0 \tag{2.7}$$

The latter relation will be used for determination of ϵ_2 inasmuch as, with reference to the folded zone, the second of Formulas (1.12) cannot be used any more.

As a result of the transformations we obtain the system

$$\frac{dx}{d\xi} = \frac{\cos\varphi}{h^2\cos\psi}, \quad \frac{dy}{d\xi} = \frac{\sin\varphi}{h^2\cos\psi}$$
$$\frac{d\varphi}{d\xi} = \frac{Q(y)}{A_* h^3 (-3\ln h)^{\mu}\cos\psi}, \quad h^2(-3\ln h)^{\mu}\xi = c \tag{2.8}$$

where c is a constant to be determined from the conditions of continuity of h at the separation line between the two zones.

We note that the same system can be obtained also immediately from the equilibrium equations of an element of the folded zone in connection with the condition (1.8), the first and the last of the relations (1.12), the first of Formulas (2.4) and the equalities (2.7), if the assumptions and simplifications are taken into account which we have indicated above.

In each actual problem, i.e. for a definite law of variation of pressure, for any given material and given initial shape and corresponding boundary conditions, we obviously can obtain the solution by means of numerical integration of the aforementioned fundamental systems and thus determine the shape and the thickness of the shell as well as the principal stresses.

The separation line between regions in the zone of extension is defined as the parallel along which $p_2 = p_1$, the separation line between the zones is determined by the condition $p_2 = 0$ applied to Equations (2.3). The continuity conditions for x, y, ϕ and h serve as matching conditions along the separation lines.

If the shell has one head only, with the origin of the coordinates located at the apex of the head (see Fig. 1), we shall have at this apex

$$x = y = \varphi = \xi = 0, \qquad h = h_0$$

Removing the indeterminacy, we find

$$\lim_{\xi \to 0} \frac{dx}{d\xi} = \lim_{\xi \to 0} \frac{x}{\xi} = h_0^{-1/2}$$

$$\lim_{\xi \to 0} \frac{d\varphi}{d\xi} = \frac{Q(0)}{2A} h_0^{-3/2} \left(-\frac{3}{2} \ln h_0\right)^{-\mu}, \qquad \lim_{\xi \to 0} \frac{dh}{d\xi} = 0$$

where h_0 must be considered as a parameter.

We note that if the shell is loaded by uniform pressure only, then $Q = qR_1/2KH_1 = \text{const}$, there is no folded zone, and the system (2.3), or the one which is its analog for the case $p_2 \geqslant p_1$, undergoes substantial simplification: it reduces to three equations for x, y, h, inasmuch as the second of Equations (2.1) is integrable by quadratures. On the basis of the latter equation we find

$$\sin \varphi = \frac{Qx}{2Ap_1 h} \tag{2.9}$$

where p_1 is expressed by the first of the two equations (2.4), or if $p_2 \geqslant p_1$ by the first of the equations (2.6).

It must be noted that even if the fundamental relationships between the stresses and deformations are considered to be valid at arbitrarily large deformations, there will always exist some value $Q = Q_{max}$ of the characteristic loading parameter which, if surpassed, makes the solution that we have suggested invalid, since it is then impossible to realize the equilibrium of the shell without violating the original assumptions. It is natural to expect that at $Q = Q_{max}$ a localization of the deformation will take place with rise of "bubbles", etc. – loss of stability of deformation in extension, analogous to the rise of a neck in a specimen in tension. Some of the simplest examples of determining Q_{max} are discussed in [8,9], without, however, taking into account the change of shape of the shell preceding localization of deformation.

We shall illustrate the considerations just presented with the example of a spherical shell acted upon by uniform internal pressure*.

Denote by ρ the ratio of the radius R of the shell after deformation to the initial radius R_1. By virtue of symmetry we must have

$$p_2 = p_1, \qquad \varepsilon_2 = \varepsilon_1 = \ln \rho$$

From these relations we derive on the basis of (1.6), (1.8) in connection with the last of Formulas (1.12) and Equation (2.9), taken with $A = 1$

$$h = \frac{1}{\rho^2}, \qquad p_1 = (3 \ln \rho)^\mu, \qquad Q = \frac{2\,(3 \ln \rho)^\mu}{\rho^3}$$

Using these formulas we can find the radius of the shell, its thickness and its principal stresses for any given loading. We easily find here also the aforementioned greatest possible loading Q_{max}, as well as the corresponding values $\rho = \rho_*$ and $h = h_*$; so we obtain

$$Q_{\max} = 2\left(\frac{\mu}{e}\right)^\mu, \qquad \rho_* = e^{\mu/3}, \qquad h_* = e^{-2\mu/3}$$

3. Shell of initially cylindrical shape. The problem undergoes here considerable simplification. We shall restrict ourselves to the study of the case that there is no folded zone. In this case we obviously have $\psi = 1/2\ \pi$ and $\xi = A$. It is convenient to take $A = 1$; then R_1 will represent the radius of the initial shape of the cylinder. The deformation is defined by the formulas

$$\varepsilon_1 = \ln\left(\frac{dx}{d\eta}\frac{1}{\cos\varphi}\right), \qquad \varepsilon_2 = \ln x, \qquad \varepsilon_3 = \ln h \tag{3.1}$$

The equilibrium equations assume the form

$$\frac{d}{dx}(xhp_1) = p_2 h, \qquad \frac{d}{dx}(xhp_1 \sin\varphi) = Q\,(y)\,x \tag{3.2}$$

where $Q(y)$ is expressed in the same way as in (2.1). Equation (2.2) remains, of course, valid. We obtain on the basis of (1.8) and (3.1) the relation

$$\frac{dx}{d\eta} = \frac{\cos\varphi}{xh} \tag{3.3}$$

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* We find in [10] an analogous problem treated for the case of linear relation between stress and deformation.

Passing to the argument $\ln x$ in the equations for the zone in tension, assumed to be solved for the derivative $dh/d\xi$, and setting for abbreviation $\ln x = \alpha$, $\ln h = \beta$, we find:

for $p_1 \geqslant p_2$

$$\frac{d\beta}{d\alpha} = -\frac{\mu + 2\alpha + \beta}{2\mu + \alpha + 2\beta} \tag{3.4}$$

with

$$p_1 = (-\alpha - 2\beta)^{\mu}, \qquad p_2 = (-\alpha - 2\beta)^{\mu - 1}(\alpha - \beta) \tag{3.5}$$

Noting that $\epsilon_2 = 0$ and consequently $p_1 = 2p_2$ when $x = 1$, at the attachment of the shell to the head, we write the solution of Equation (3.4) in the form

$$\alpha^2 + (\beta + \mu)\alpha + (\beta + \mu)^2 = (\beta_0 + \mu)^2 \tag{3.6}$$

where $\beta_0 = \ln h_0$ represents the value of $\ln h$ at $x = 1$.

For $p_2 \geqslant p_1$ we have

$$\frac{d\beta}{d\alpha} = \frac{\mu(\alpha + 2\beta) - 3\beta + (\alpha - \beta)(2\alpha + \beta)}{\mu(\alpha + 2\beta) - 3\alpha - (\alpha - \beta)(\alpha + 2\beta)} \tag{3.7}$$

with

$$p_1 = -(\alpha + 2\beta)(\alpha - \beta)^{\mu - 1}, \qquad p_2 = (\alpha - \beta)^{\mu} \tag{3.8}$$

Equation (3.7) can be integrated numerically.

We see that independently of the law of variation of pressure along the height, the fundamental system separates, the dimensionless thickness of the shell and the dimensionless principal stresses, expressed in terms of the radial coordinate, depend only on the parameter μ, which characterizes the material, and the parameter h_0, which takes care of the influence of all other factors (ratio of the dimensions, their absolute values, the pressure characteristics, etc.).

Equations (3.4) and (3.7) are equivalent, respectively, to the corresponding equations derived in [4]; the integral curves $h(x)$, derived there and plotted for $\mu = 1/3$, can therefore be directly used in this case even if $Q \neq$ const.

We note that if use is made of the relationship (1.9), then the fundamental system for shells of initially cylindrical shape separates too. On the basis of (1.10), (1.8), (1.11), (3.1) and of the first of Equations (3.2), we obtain, instead of (3.4) and (3.7), one equation (since the relationship between p_1 and p_2 is of no significance):

$$\frac{d\beta}{d\alpha} = -\frac{\mu(\alpha + 2\beta)(2\alpha + \beta) - 3\alpha\beta + 2(2\alpha + \beta)(\alpha^2 + \beta^2 + \alpha\beta)}{\mu(\alpha + 2\beta)^2 + 3\alpha^2 + 2(\alpha + 2\beta)(\alpha^2 + \beta^2 + \alpha\beta)}$$

This equation, too, can be integrated numerically, setting $\beta = \beta_0 = \ln h_0$ at $\alpha = 0$.

Having the curves $h(x)$, we can obtain the complete solution in a majority of actual problems by means of simple operations.

On the basis of the second of Equations (3.2) we obtain

$$\sin\varphi = \frac{1}{xh\,p_1}\int_1^x Q(y)\,x\,dx \qquad (3.9)$$

This equation must be treated in conjunction with Equation (2.2), and the solution is obtained with the aid of Formulas (3.5) or (3.8) for p_1. At the same time η is determined by means of the quadrature

$$\eta = \int_1^x \frac{xh}{\cos\varphi}\,dx \qquad (3.10)$$

which follows from (3.3).

In the case of constant pressure the system (3.9) to (2.2) separates too, and y is determined by quadrature. Let us consider the case of pressure linearly varying with Y. Let

$$q(Y) = k(B - Y)$$

where k and B are given constants. We set

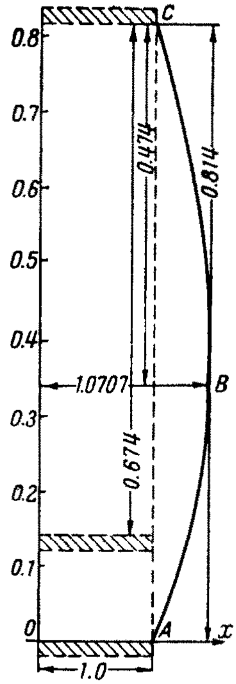

Fig. 2.

$$\frac{B}{R_1} = b, \quad \frac{kR_1B}{2KH_1} = Q_0$$

Then

$$Q(y) = \frac{Q_0}{b}(b - y)$$

and on the basis of (3.9), (2.2) and (3.10) we find with the use of the argument y, which is more convenient for the computations

$$\sin\varphi = \frac{Q_0}{2bp_1 h}\left[x(b-y) + \frac{1}{x}\int_0^y x^2 dy\right], \quad \frac{dx}{dy} = \cot\varphi, \qquad \eta = \int_0^y \frac{xh}{\sin\varphi}\,dy \quad (3.11)$$

The system (3.11) can be solved numerically very simply. It should be noted, however, that as in the problem with q = const, the parameters h_0, b and Q_0 cannot be prescribed arbitrarily. For each type of problem we have first to find limits for possible values of these parameters. Reference [4] indicates in a detailed manner the method for determination of such limits.

We give below the results of the solution of the equilibrium problem for a shell fixed along the periphery of one of its heads; the shell is acted upon by pressure linearly varying along its height. The computations were carried out for the cases $b = 2$ and $b = 3$ with the selected values $Q_0 = 1.12$ and $h_0 = 0.8$ at $\mu = 1/3$.

Figure 2 shows the shape, assumed by the shell, and its characteristic dimensions (referred to R_1) in the first case; these dimensions are the initial length l_1, the final length l, the radius x_B of the largest parallel circle and the distance $l - y_B$ of the latter from the upper fixed head.

For the case $b = 3$, the following values have been computed:

$$l_1 = 0.561, \quad l = 0.675, \quad x_B = 1.069, \quad l - y_B = 0.375.$$

The values of the dimensionless stresses and thickness at the points *A, B, C* (Fig. 2) are given in the following table:

Points	$b = 2$			$b = 3$		
	p_1	p_2	h	p_1	p_2	h
A and *C*	0.764	0.382	0.8	0.764	0.382	0.8
B	0.784	0.559	0.760	0.783	0.554	0.761

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